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The Luttinger Model

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The thermodynamic correlation functions of the Luttinger model are computed. The main tool is a precise bosonization formula for the fermion field.

KEY WORDS: Thermodynamic correlation functions; two-dimensional models; Luttinger model; relativistic fermion fields; Thirring model; boson–fermion reciprocity; bosonization; collective coordinates.

1. INTRODUCTION

This paper is organized as follows. In Section 2 we review the history of the Luttinger model. In Section 3 the model is described more precisely and the derivation of thermodynamic correlation functions is given. The latter part has been announced previously in Ref. 28.

Additional material is contained in several appendices. In Appendix A we remind the reader of the relation between the massive, two-dimensional, nonrelativistic and the massless, relativistic fermion field. Appendix B contains a proof of the bosonization formula for the free, massless fermion field. Technicalities pertinent to the derivation of the thermodynamic correlation functions are contained in Appendix C. In Appendix D several examples of correlation functions are discussed.

2. ON THE HISTORY OF THE LUTTINGER MODEL

The dynamics of the Luttinger model[1] is defined in terms of the Hamiltonian

$$H = \int_0^L dx :\psi^* i\, \partial_0 \psi:(x) + \int_0^L dx\, dy\, j_\mu(x) V(x-y) j^\mu(y) \tag{2.1}$$

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ψ is a two-component relativistic fermion field in one space and one time dimension, L is the length of the periodic box, j_μ is the current of the fermion field, and V is the interaction potential. If the length is infinite and the interaction strictly local, i.e., $V(x) = \delta(x)$, the Hamiltonian defines the Thirring model.[2,18,19] Of course, some care is required in order to make sense out of all the quantities involved, such as composite fields, thermodynamic limit, etc. In this section, however, we will not be concerned with such questions.

The literature on the Luttinger model up to 1965 was discussed by Lieb and Mattis.[4] The historical development up to 1964 of two-dimensional models of quantum field theory, in particular of the Thirring model, is reviewed in the Cargèse lectures of Wightman.[17] We will sketch this period briefly and then mention some more recent work.

The original motivation for the study of the Thirring and Luttinger models was to obtain information about interacting fermion systems. When Tomonaga[3] in 1950 proposed his model for massive, nonrelativistic fermions in one space dimension he made some assumptions to make his model more manageable. Luttinger then proposed a Hamiltonian which already contains those approximations from the beginning. The nature of the approximations makes it clear that the spin degree of freedom in the Thirring model corresponds in the Luttinger model to the direction of the momentum of the fermions. For details on the relation of the Tomonaga model to the Luttinger model we refer to Chapter 4 of the book by Lieb and Mattis[4] (see also Ref. 5). At that time it was expected that systems with many fermions would exhibit bound states, collective modes or plasmons, in terms of which the model could be described more easily. In fact this turned out to be correct and provided the key to the solution of the Luttinger model and the solution of the Thirring model as well.

The Luttinger model was solved by Lieb and Mattis.[6] The solution showed interesting structures, e.g., a nontrivial spectrum. Hence it attracted much attention and proved to be an excellent laboratory for checking the validity of approximations and testing new ideas. Thus it became clear that perturbation theory does in general give a wrong answer, e.g., to the question of whether there is a Fermi surface or not,[6] and that a random phase approximation gives good results for the correlation functions and the spectrum but incorrect results for the momentum distribution function.[7] For field theory the Thirring model was particularly useful because it permitted a thorough analysis of operator product expansions.[8,9]

Generalizations of the Luttinger model were being discussed soon after Luttinger's original article. In 1965 Mattis[10] and Overhauser[11] introduced an additional spin degree of freedom into the model. In field-theoretic language this corresponds to an additional $SU(2)$ isospin. For the rather special interaction considered by Mattis and Overhauser the model was still soluble because it could be translated into two independent Luttinger models and exhibits two types of collective modes, plasmons and magnons. The former are related to density fluctuations, the latter to spin density fluctuations. If all possible fermion interactions are included, the model cannot be solved explicitly anymore since it is equivalent to an ordinary Luttinger model and an independent Luttinger model with a mass term.[12] If the coupling parameters are chosen so that in the second Luttinger model the current–current coupling vanishes, the model is again soluble. This was pointed out by Emery and Luther in 1974.[13] For general coupling constants the spectrum can be discussed using WKB methods.[12] An analysis of this model, called the backward scattering model, has been carried through by Heidenreich.[14] In this work particular care was given to the charge structure of the fermions. In field theory the analogous model is the Thirring model with $SU(2)$ isospin. The slightly more general model with an $SU(n)$, n an integer, internal degree of freedom was used to analyze renormalization group equations.[15] Recently the Thirring model with an internal $SU(2)$ degree of freedom, respectively the Thirring model with mass, received much attention because the fermions in this model can be put into correspondence with the quantum soliton of a boson field theory where the fundamental field is a solution of the Sine–Gordon equation.[16]

The historical development leading to the solution of the Luttinger and Thirring models was long and complicated.[4,17] Three methods turned out to be successful: In the first one—used by Johnson[18] in 1961 for the Thirring model—Ward identities are used to compute the fermion n-point function. It was not clear, however, whether the n-point function so computed is coming from a quantum field. The solution was therefore incomplete. In 1965 Lieb and Mattis[6] gave an operator solution of the Luttinger model. They computed the free energy, the susceptibility, and the momentum distribution function at zero temperature. The key step in their method was to notice that: (1) The Hamiltonian can effectively be replaced by a quadratic expression in boson collective modes and charge operators; in their analysis they rediscovered Kronig's identity. (2) This identity expresses the free fermion Hamiltonian $H_0{}^{\mathrm{F}}$ in terms of the boson Hamiltonian $H_0{}^{\mathrm{B}}$ and the charges Q^+ and Q^- of the fermions moving in the plus and the minus directions:

$$H_0{}^{\mathrm{F}} = H_0{}^{\mathrm{B}} + (\pi/L)(Q_+{}^2 + Q_-{}^2) \tag{2.2}$$

Since the interaction term in H is quadratic in the boson field too, the total Hamiltonian can now be diagonalized by a Bogoliubov–Valatin transformation.

Kronig's identity dates back to the neutrino theory of light[21] and holds as an operator identity.[22] The mystery over this identity is possibly reduced by the remark that in any system of noninteracting, massless particles in one space and one time dimension there are massless bound states.[23]

In 1968 Johnson's partial solution of the Thirring model was completed by Klaiber.[19] He constructed a quantum field giving rise to Johnson's n-point functions. His method differs from those already mentioned. He starts from classical field theory and reinterprets the solution of the classical field equation as a quantum field in Fock space of a free fermion field. The model can also be solved by the methods of Lieb and Mattis.[6] However, in the case of the Thirring model an ultraviolet renormalization is necessary.[33]

The method of Lieb and Mattis was used to compute the zero-temperature, fermion two-point function,[24,25] and Johnson's method was used to get the zero-temperature n-point functions for the Luttinger model.[26] Furthermore, the fermion two-point function for arbitrary temperature was found by the same method.[27]

Finally, the method of Lieb and Mattis complemented by the boson–fermion reciprocity formula, Eq. (B22) of Appendix B, was used to compute the n-point function of the Luttinger model for arbitrary temperature. This result was announced in Ref. 28 and its derivation will be the main subject of the following section.

The boson–fermion relations give an explicit operator expression for the fermion field in terms of the boson field (collective modes), charges Q_+ and Q_-, and charge shift operators U_+ and U_- (defined later). It has been used in one form or another by many authors, in particular, by Schotte,[29] Lowenstein and Swieca,[30] Dell'Antonio *et al.*,[31] and Mandelstam.[32] We will use the formulation given in Ref. 33.

The possibility of a (polar coordinate type) generalization of the boson–fermion relation to higher dimensions has been noted repeatedly in talks by one of us (D.A.U.) during the last few years. We give a formulation of this idea at the end of Appendix B. This has independently been noted by A. Luther.[4]

The boson–fermion relation clarifies not only the aforementioned relation of the Luttinger model to the Sine–Gordon model, but also its relation to the classical Coulomb gas.[34]

3. THE LUTTINGER MODEL AND ITS THERMODYNAMIC n-POINT FUNCTIONS

The Luttinger model[1] deals with the thermodynamics and statistical mechanics of a self-interacting, one-dimensional Fermi system enclosed in a periodic box of length $0 < L < \infty$. To set up this model, we assume as

[4] Winter Workshop les Houches (1978) and Nordita Preprint (1978).

given an irreducible Fock system of particle (and hole) creation and annihilation operators $\alpha_p^*(\kappa)$ and $\alpha_p(\kappa)$ [and $\alpha_h^*(\kappa)$ and $\alpha_h(\kappa)$] acting on Fock space $\mathscr{F}$ with vacuum Ω. These operators thus satisfy the usual canonical anticommutation relations (CAR) together with

$$\alpha_p(\kappa)\Omega = 0 \qquad \text{and} \qquad \alpha_h(\kappa)\Omega = 0 \tag{3.1}$$

For technical reasons it is convenient to shift the basic momenta in the discrete momentum spectrum $K_L := (2\pi/L)\mathbb{Z}$ by an amount π/L to obtain $K_L' = (\pi/L)\mathbb{Z}_{\text{odd}}$. Thus in all formulas involving the $\alpha_p(\kappa)$ and $\alpha_h(\kappa)$ the variable κ assumes its values in K_L'.

The basic fixed-time Fermi field depending on the space coordinate x, with its helicity components ψ_+ and ψ_-, is defined as

$$\psi_\pm(x) = \frac{1}{\sqrt{L}} \sum_{\pm\chi>0} [e^{i\chi x}\alpha_p(\chi) + e^{-i\chi x}\alpha_h^*(\chi)] \tag{3.2}$$

and is seen to satisfy the anticommutation relations

$$\begin{aligned} \{\psi_\pm(x), \psi_\pm(y)\} &= 0 \\ \{\psi_\pm(x), \psi_\pm^*(y)\} &= (1/L) \sum_\chi e^{i\chi(x-y)} \end{aligned} \tag{3.3}$$

In terms of the γ-matrices for the metric with signature $(+, -)$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.4}$$

the Fermi current and pseudocurrent are given by

$$j^\mu := :\overline{\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}} \gamma^\mu \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}:; \qquad j^{\mu 5} := :\overline{\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}} \gamma^\mu\gamma^5 \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}: \tag{3.5}$$

with : : indicating Fermi normal ordering, while the corresponding charge and pseudocharge are

$$Q = \int_0^L dx\, j^0(x) \qquad \text{and} \qquad Q^5 = \int_0^L dx\, j^{05}(x) \tag{3.6}$$

In what follows it will be more convenient to employ the associated "light-cone combinations"

$$j_\pm := \tfrac{1}{2}(j^0 \pm j^1) = :\psi_\pm^*\psi_\pm: \tag{3.7}$$

$$Q_\pm := \int_0^L dx\, j_\pm(x) \tag{3.8}$$

Clearly, this means that

$$j^0 = j^{15} = j_+ + j_-; \qquad j^1 = j^{05} = j_+ - j_-$$
$$Q = Q_+ + Q_-; \qquad Q^5 = Q_+ - Q_- \tag{3.9}$$

A more concise notation as well as more calculational details of much of the following are to be found in Appendices B and C, to which we refer the interested reader.

If we introduce the Fourier decomposition of the current densities via

$$j_\pm(x) = (1/L) \sum_{p \in K_L} e^{ipx} \tilde{j}_\pm(p) \tag{3.10}$$

then $Q_\pm = \tilde{j}_\pm(0)$. A careful evaluation of the occurring infinite sums on the domain $\mathscr{D}$, consisting of polynomials in the $\alpha_p{}^*(\kappa)$ and $\alpha_h{}^*(\kappa)$ applied to Ω, yields the basic commutation relations

$$[\tilde{j}_\pm(p), \tilde{j}_\pm(-p)] = \pm(L/2\pi)P \tag{3.11}$$

with all other commutators vanishing.[21]

In particular, the charges $Q_\pm$ commute with each other and all the $\tilde{j}_\pm(p)$. We conclude from Eq. (3.11) that the combinations (for $p > 0$ in K_L)

$$c(\pm p) \coloneqq i(2\pi/Lp)^{1/2} \tilde{j}_\pm(\pm p)$$
$$c^*(\pm p) \coloneqq -i(2\pi/Lp)^{1/2} \tilde{j}_\pm(\mp p) \tag{3.12}$$

are formally adjoint and satisfy the canonical commutation relations (CCR) on $\mathscr{D}$

$$[c(q), c^*(q')] = \delta_{q,q'} \qquad \text{for} \quad q \neq 0 \neq q' \tag{3.13}$$

with all other commutators vanishing.

The nature of the representation of the CCR involved here was analyzed in Ref. 22. It was shown that the commuting self-adjoint charges $Q_\pm$ both have spectrum consisting of the integers $\mathbb{Z}$. As the $c(q)$ and $c^*(q')$ commute with $Q_\pm$, each joint spectral subspace of $Q_\pm$, indexed by the eigenvalues $n_\pm \in \mathbb{Z}$, reduces this canonical system. It was found that the corresponding restrictions of the $c(q)$ and $c^*(q')$ to each of these subspaces constituted irreducible Fock systems of the CCR with vacua Ω_{n_-,n_+}.

A specific choice of $\Omega_{n_-,n_+} \in \mathscr{F}$ is exhibited in Eq. (B12) of Appendix B. The presence of this representation of the CCR suggests the introduction of the following Bose fluctuation field, which describes deviations from the vacua Ω_{n_-,n_+}:

$$\varphi_\pm(x) \coloneqq \sum_{\pm p > 0} (2\pi/L|p|)^{1/2} [e^{ipx} c(p) + e^{-ipx} c^*(p)] \tag{3.14}$$

From Eqs. (3.12) and (3.14) we easily deduce that

$$\varphi_\pm(x) = \sum_{k\neq 0} (\pm 2\pi i/Lk)e^{ikx}j_\pm(k) \tag{3.15}$$

$$j_\pm(x) = (1/L)Q_\pm \mp (1/2\pi)(d/dx)\varphi_\pm(x) \tag{3.16}$$

The technical device that permits the complete calculation of the thermodynamic correlation functions of the Luttinger model consists of a precise formula, the "bosonization formula," giving the field $\psi_\pm$ in terms of the field $\varphi_\pm$ and the charge structure. This formula is derived in all detail in Appendix B. Here we only present the outline of the argument and refer the reader to that appendix for the full story.

According to formula (B18), these two fields are related by

$$[\psi_\pm(x), \varphi_\pm(y)] = -i\psi_\pm(x)[\varphi_\pm(x), \varphi_\pm(y)] \tag{3.17}$$

where the second factor on the right side is scalar. On the other hand, since the $\psi_\pm$ field lowers the $Q_\pm$ charge by one unit, we obtain an expression which commutes with the charges by using $U_\pm{}^*\psi_\pm(x)$ in formula (3.17) instead of $\psi_\pm(x)$. Here U_+ (resp. U_-) is the unique unitary on $\mathscr{F}$ that commutes with all the $\varphi_\pm(x)$ and maps Ω_{n_-,n_+} to Ω_{n_-,n_+-1} (resp. Ω_{n_--1,n_+}). We also let $U_\pm{}^*$ be the corresponding adjoints, which therefore raise the respective charge by one unit. This modified equation (3.17) is to be compared with the easily derived formula

$$[\vdots e^{-i\varphi_\pm}\vdots(x), \varphi_\pm(y)] = -i\vdots e^{-i\varphi_\pm}\vdots(x)[\varphi_\pm(x), \varphi_\pm(y)] \tag{3.18}$$

where $\vdots\ \vdots$ indicates Bose normal ordering.

The irreducibility of the Bose Fock components of the $\varphi_\pm$ in each charge sector of $\mathscr{F}$ then allows one to conclude that $U_\pm{}^*\psi_\pm(x)$ differs from $\vdots e^{i\varphi_\pm}\vdots(x)$ by a function $F_\pm(x)$ depending only on the charge eigenvalues of each sector. Its explicit form is exhibited in Eq. (B21), and Theorem 1 in Appendix B gives the *bosonization formula* for the field $\psi_\pm$ in the form

$$\psi_\pm(x) = U_\pm F_\pm(x)\vdots e^{-i\varphi_\pm}\vdots(x) \tag{3.19}$$

with an analogous expression valid for the adjoint field.

We next turn to the specification of the interaction for the Luttinger model. Its Hamiltonian H^F is formally given as

$$H^F = \int_0^L dx\ {:}\psi^* i\,\partial_t\psi{:}(x) + \int_0^L dx \int_0^L dy\, j_\mu(x)V(x-y)j^\mu(y) \tag{3.20}$$

The interaction potential V will be restricted to satisfying the following conditions, formulated in terms of its Fourier transform:

$$V(x) = (1/L) \sum_{k \in K_L} e^{-ikx}(\pi/2) W_k \tag{3.21}$$

$$W_k = (W_k)^* = W_{-k} \qquad \text{(reality and evenness)} \tag{3.22}$$

$$W_0 = 0; \qquad -1 < W_k < 1 \tag{3.23}$$

$$\sum_k |k| W_k^2 < \infty \tag{3.24}$$

Now the crucial ingredient for the bosonization of the Fermi Hamiltonian H^{F} is the bosonization of the kinetic energy part provided by the *Kronig identity* (see Ref. 20):

$$\int_0^L dx :\psi^* i\, \partial_t \psi:(x) = (\pi/L)(Q_+^2 + Q_-^2) + \sum_{p \neq 0} |p| c^*(p) c(p) \tag{3.25}$$

Using Eqs. (3.15), (3.21), and (3.25) in Eq. (3.20) one easily derives the identity

$$H^{\mathrm{F}} = (\pi/L)(Q_+^2 + Q_-^2) + \sum_{p>0} H_p \tag{3.26}$$

where

$$\begin{aligned} H_p = p[&(c^*(p)c(p) + c^*(-p)c(-p)) \\ &- W_p(c^*(p)c^*(-p) + c(-p)c(p))] \end{aligned} \tag{3.27}$$

The assumptions (3.22) and (3.23) ensure that there is a unique real $\lambda_p = \lambda_{-p}$ with

$$W_p = \tanh(2\lambda_p) \tag{3.28}$$

In terms of these λ_p we define the unitary operators

$$U_p := \exp\{\lambda_p[c(p)c(-p) - c^*(p)c^*(-p)]\} \tag{3.29}$$

An easy calculation then shows that the Bogoliubov–Valatin transformation generated by U_p is

$$U_p \begin{pmatrix} c(-p) \\ c^*(p) \end{pmatrix} U_p^* = \begin{pmatrix} \cosh \lambda_p & \sinh \lambda_p \\ \sinh \lambda_p & \cosh \lambda_p \end{pmatrix} \begin{pmatrix} c(-p) \\ c^*(p) \end{pmatrix} \tag{3.30}$$

It will be useful to employ the abbreviations

$$\begin{aligned} h_p^+ &:= \cosh \lambda_p = [\tfrac{1}{2} + \tfrac{1}{2}(1 - W_p^2)^{-1/2}]^{1/2} \\ h_p^- &:= \sinh \lambda_p = [-\tfrac{1}{2} + \tfrac{1}{2}(1 - W_p^2)^{-1/2}]^{1/2} \operatorname{sgn} W_p \end{aligned} \tag{3.31}$$

Equation (3.30) together with the choice of Eq. (3.28) applied to the H_p of Eq. (3.27) shows that H_p is diagonalized by this canonical transformation to give

$$U_p H_p U_p^* = \omega_p[c^*(p)c(p) + c^*(-p)c(-p)] - \eta_p \tag{3.32}$$

where the "quasiparticle energies" are

$$\omega_p = |p| \operatorname{sech}(2\lambda_p) = |p|(1 - W_p{}^2)^{1/2} \tag{3.33}$$

while the "energy shift" is

$$\eta_p = |p| - \omega_p = |p|[1 - (1 - W_p{}^2)^{1/2}] \geqslant 0 \tag{3.34}$$

Due to the assumption (3.24), $\eta = \sum_{p>0} \eta_p$ converges to a finite, nonnegative value and

$$U = \prod_{p>0} U_p \tag{3.35}$$

converges (strongly) to a unitary operator on $\mathscr{F}$, which diagonalizes the Luttinger Hamiltonian H^{F} to give

$$H \coloneqq UH^{\mathrm{F}}U^* = (\pi/L)(Q_+{}^2 + Q_-{}^2) + \sum_{p\neq 0} \omega_p c^*(p)c(p) - \eta \tag{3.36}$$

We now introduce new space- and time-dependent Fermi and Bose fields by setting

$$\Psi_\pm(t, x) \coloneqq e^{iHt}U\psi_\pm(x)U^*e^{-iHt} \tag{3.37}$$

$$\Phi_\pm(t, x) \coloneqq e^{iHt}U\varphi_\pm(x)U^*e^{-iHt} \tag{3.38}$$

Since $\Psi_\pm(t, x)$ is the U transform of the interacting Luttinger field $\exp(iH^{\mathrm{F}}t)\psi_\pm(x)\exp(-iH^{\mathrm{F}}t)$, it will give the same thermodynamic correlation functions as this latter field. The explicit representation of $\Phi_\pm$ in terms of the $c(q)$ and $c^*(q)$ operators is provided by Eq. (C3) of Appendix C.

Applying the $e^{iHt}U$ transformation to the bosonization formula (3.19), we find that

$$\Psi_\pm(t, x) = U_\pm F_\pm(x \mp t) \vdots e^{-i\Phi_\pm} \vdots (t, x) e^{-A} \tag{3.39}$$

where the factor e^{-A} results from boson Wick ordering and the constant

$$A = (\pi/L) \sum_{p>0} (\omega_p^{-1} - p^{-1}) \tag{3.40}$$

is finite because of assumption (3.24).

In terms of these new Fermi fields the quantities of interest are the vacuum expectation values

$$\langle \Omega, \Psi_\tau{}^\sigma(\mathbf{t}, \mathbf{x})\Omega \rangle \tag{3.41}$$

and the thermodynamic correlation functions

$$\langle\langle \Psi_{\tau}^{\sigma}(\mathbf{t}, \mathbf{x})\rangle\rangle = \mathrm{Tr}_{\mathscr{F}}[e^{-\beta H}\Psi_{\tau}^{\sigma}(\mathbf{t}, \mathbf{x})]/\mathrm{Tr}_{\mathscr{F}}\, e^{-\beta H} \tag{3.42}$$

where $\Psi_{\tau}^{\sigma}(\mathbf{t}, \mathbf{x})$ stands for an arbitrary product of finitely many $\Psi_{\pm}(t_j, x_j)$ and $\Psi_{\pm}^*(t_j, x_j)$ factors.

In other words, as explained at the beginning of Appendix B and Eqs. (C6)–(C8), if n is a nonnegative integer and t_j and x_j for $i \leqslant j \leqslant n$ are real numbers, σ_j and τ_j are elements of $S = \{+, -\} = \{+1, -1\}$, then

$$\Psi_{\tau}^{\sigma}(\mathbf{t}, \mathbf{x}) = \prod_{j}^{1,n} \Psi_{\tau_j}^{\sigma_j}(t_j, x_j) \qquad \text{[Eq. (C7)]}$$

with

$$\Psi_{\tau_j}^{-}(t_j, x_j) = \Psi_{\pm}(t_j, x_j) \qquad \text{if} \quad \tau_j = \pm$$

$$\Psi_{\tau_j}^{+}(t_j, x_j) = \Psi_{\pm}^{*}(t_j, x_j) \qquad \text{if} \quad \tau_j = \pm$$

It follows from the charge structure of the expressions (3.41) and (3.42) that they vanish identically unless the number of Ψ_+ (resp. Ψ_-) factors in Ψ_{τ}^{σ} equals the number of Ψ_+^* (resp. Ψ_-^*) factors. This happens precisely if $\sum_j \sigma_j = 0$ and $\sum_j \sigma_j\tau_j = 0$. Accordingly, we shall assume from now on, without explicit mention, that these conditions are satisfied for Ψ_{τ}^{σ}.

As shown in Appendix C, the relevant Bose expression for the calculation is [see Eq. (C8)]

$$\begin{aligned} \Phi_{\tau}^{\sigma}(\mathbf{t}, \mathbf{x}) &\coloneqq \sum_{j}^{1,n} \sigma_j \Phi_{\tau_j}(t_j, x_j) \\ &= \sum_{\mathbf{q} \neq 0} [\epsilon_{\mathbf{q}\tau}^{\sigma}(\mathbf{t}, \mathbf{x}) c(\mathbf{q}) + \epsilon_{\mathbf{q}\tau}^{*\sigma}(\mathbf{t}, \mathbf{x}) c^*(\mathbf{q})] \end{aligned}$$

with

$$\epsilon_{\mathbf{q}\tau}^{\sigma}(\mathbf{t}, \mathbf{x}) \coloneqq \left(\frac{2\pi}{L|\mathbf{q}|}\right)^{1/2} \sum_{j}^{1,n} \sigma_j \{\exp[-i(\omega_{\mathbf{q}} t_j - \mathbf{q} x_j)]\} h_{\mathbf{q}}^{\tau_j \operatorname{sgn} \mathbf{q}}$$

The charge structure under the above assumption on Ψ_{τ}^{σ} is reflected in [Eq. (C10)]

$$Q_{\tau}^{\sigma}(\mathbf{t}, \mathbf{x}) = (2\pi/L) \sum_{j}^{1,n} \sigma_j (t_j - \tau_j x_j) Q_{\tau_j}$$

and

$$\prod_{j}^{1,n} U_{\tau_j}^{\sigma_j} = 1_{\mathscr{F}}$$

In terms of this notation the bosonization formula implies that [Eq. (C14)]

$$\Psi_\tau^\sigma(\mathbf{t},\mathbf{x}) = \langle\Omega, \Psi_\tau^\sigma(\mathbf{t},\mathbf{x})\Omega\rangle \exp(iQ_\tau^\sigma)(\mathbf{t},\mathbf{x}) \vdots\exp(i\Phi_\tau^\sigma)\vdots(\mathbf{t},\mathbf{x})$$

and [Eq. (C15)]

$$\begin{aligned}
&\log\langle\Omega, \Psi_\tau^\sigma(\mathbf{t},\mathbf{x})\Omega\rangle \\
&= \sum_j^{1,n}\left[\frac{i\pi}{L}(t_j - \tau_j x_j) - \tfrac{1}{2}\log L - A\right] \\
&\quad + \sum_{j<k}^{1,n} \sigma_j\sigma_k\left[i\pi\sigma_j\,\delta_{\tau_j,+} + i\frac{2\pi}{L}(t_j - \tau_j x_j)\,\delta_{\tau_j,\tau_k} - A_{\tau_j,\tau_k}(t_j - t_k, x_j - x_k)\right]
\end{aligned}$$

with A_{τ_j,τ_k} given by Eq. (C12).

Thus, in view of Eqs. (3.26) and (C14) we find

$$\begin{aligned}
&\mathrm{Tr}_{\mathscr{F}}[e^{-\beta H}\Psi_\tau^\sigma(\mathbf{t},\mathbf{x})]e^{-\beta n} \\
&\quad = \langle\Omega, \Psi_\tau^\sigma(\mathbf{t},\mathbf{x})\Omega\rangle \\
&\qquad \times \mathrm{Tr}_{Q_\pm}\left\{\exp\left[-\beta\frac{\pi}{L}(Q_+{}^2 + Q_-{}^2)\right]\exp[iQ_\tau^\sigma(\mathbf{t},\mathbf{x})]\right\} \\
&\qquad \times \mathrm{Tr}_{\mathrm{B}}\left\{\exp\left[-\beta\sum_{\mathbf{q}\neq 0}\omega_{\mathbf{q}}c^*(\mathbf{q})c(\mathbf{q})\right]\vdots\exp(i\Phi_\tau^\sigma)\vdots(\mathbf{t},\mathbf{x})\right\}
\end{aligned}$$

where the charge trace and the Bose sector trace can be computed explicitly in terms of Jacobian theta functions and the confluent hypergeometric function, respectively, as shown in Eqs. (C16)–(C23). The special case $\Psi_\tau^\sigma = 1_{\mathscr{F}}$ gives the expression (C24) for $\mathrm{Tr}_{\mathscr{F}}\, e^{-\beta H}$, which after division according to Eq. (3.42) yields the final result:

$$\begin{aligned}
&\log\langle\langle\Psi_\tau^\sigma(\mathbf{t},\mathbf{x})\rangle\rangle - \log\langle\Omega, \Psi_\tau^\sigma(\mathbf{t},\mathbf{x})\Omega\rangle \\
&\quad = 4\sum_{n=1}^{\infty}\frac{(-)^n}{n}\frac{e^{-\beta(\pi/L)n}}{1 - e^{-\beta(2\pi/L)n}} \\
&\qquad \times\left\{\sin^2\left[n\frac{\pi}{L}\sum_{\tau_j=+}\sigma_j(x_j - t_j)\right] + \sin^2\left[n\frac{\pi}{L}\sum_{\tau_j=-}\sigma_j(x_j + t_j)\right]\right\} \\
&\qquad - \sum_{\mathbf{q}\neq 0}\frac{\exp(-\beta\omega_{\mathbf{q}})}{1 - \exp(-\beta\omega_{\mathbf{q}})}|\epsilon_{\mathbf{q}\tau}^\sigma(\mathbf{t},\mathbf{x})|^2 \qquad \text{[Eq. (C27)]}
\end{aligned}$$

APPENDIX A. ON THE RELEVANCE OF TWO-DIMENSIONAL RELATIVISTIC FERMION FIELDS FOR SOLID STATE PHYSICS

In the following we consider a box of length $0 < L < \infty$. We prove that the Hamiltonian $H_0(\phi)$ of a free, nonrelativistic fermion field ϕ with mass

$m > 0$ and spin S is approximated by the Hamiltonian $H_0(\psi)$ of a free, relativistic fermion field ψ with mass $m = 0$, spin $\frac{1}{2}$, and isospin S.

Definition 1. $\phi(t, x)$ can be written as

$$\phi(t, x) := (1/\sqrt{L}) \sum_{k,s} e^{-i(\omega_k t - kx)} a(k, s) \tag{A1}$$

with $\omega_k := k^2/2m$.

Remark 2. (a) Without further notice it will be assumed that the fermionic momenta take their values in $K_L' = (\pi/L)\mathbb{Z}_{\text{odd}}$.

(b) The operators $a(k, s)$ obey the CAR.

(c) ϕ and $H_0(\phi) = \sum_{k,s} \omega_k a^*(k, s)a(k, s)$ are operators on Fock space F with vacuum $\Omega(a)$.

(d) From now on we suppress all spinor indices.

Definition 3. $\psi(t, x)$ can be represented as

$$\psi(t, x) := (1/\sqrt{L}) \sum_{k} [\exp(-i\mathbf{kx})]a(k) \tag{A2}$$

with $\mathbf{kx} = |k|t - kx$.

Remark 4. (a) The operators

$$a(k) = \begin{pmatrix} a_-(k) \\ a_+(k) \end{pmatrix}$$

obey the CAR, where all isospin indices have been omitted.

(b) ψ and

$$H_0(\psi) = \sum_{\tau=\pm} \sum_{k} \tau k a_\tau^*(k) a_\tau(k)$$

are operators on the Fock space $F \otimes F$ with vacuum $\Omega(a_-) \otimes \Omega(a_+)$.

Definition 5. To construct a unitary transformation from a subspace of F to a subspace of $F \otimes F$ we define the following projection operators: On F

$$P(\epsilon) := \sum_{\tau} P_\tau(\epsilon) \tag{A3}$$

with

$$P_\tau(\epsilon) := \sum_{|k - \tau k_F| < \epsilon} a^*(k)a(k)$$

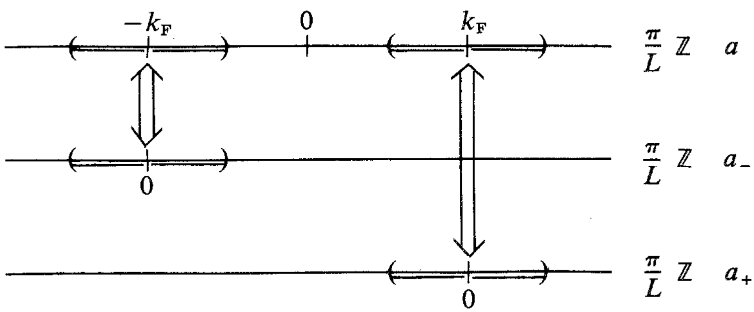
On $F \otimes F$

$$Q(\epsilon) := \sum_{\tau} Q_\tau(\epsilon) \tag{A4}$$

with

$$Q_\tau(\epsilon) := \sum_{|k|<\epsilon} a_\tau^*(k)a_\tau(k)$$

where $\tau \in \{+, -\}$, $k \in K_L'$, and $k_F \in K_L := (2\pi/L)\mathbb{Z}$ denotes the Fermi momentum.



Obviously the following statement holds:

Lemma 6. For $\epsilon > 0$ sufficiently small, the operator J, defined by

$$J: \qquad \Omega(a) \mapsto \Omega(a_-) \otimes \Omega(a_+)$$

$$J: \quad a(\tau k_F + k) \mapsto a_\tau(k) = Ja(\tau k_F + k)J^{-1}$$

generates an isomorphism from

$$P(\epsilon)F \to Q(\epsilon)F \otimes F$$

Remark 7. In the following we identify both subspaces.

Theorem 8. Let

$$H(\phi, \epsilon) := \sum_\tau P_\tau(\epsilon)H_0(\phi)P_\tau(\epsilon)$$

$$= \sum_\tau \sum_{|k-\tau k_F|<\epsilon} \omega_k a^*(k)a(k) \tag{A5}$$

and

$$H(\psi, \epsilon) := \sum_\tau Q_\tau(\epsilon)H_0(\psi)Q_\tau(\epsilon)$$

$$= \sum_\tau \sum_{|k|<\epsilon} \tau k a_\tau^*(k)a_\tau(k) \tag{A6}$$

be the restrictions of $H_0(\phi)$ and $H_0(\psi)$ to the subspaces $P(\epsilon)F$ and $Q(\epsilon)F \otimes F$, respectively; then

$$H(\phi, \epsilon) = \frac{k_F}{m} H(\psi, \epsilon) + \frac{k_F^2}{2m} Q(\epsilon) + R \tag{A7}$$

with

$$\|R\| \leqslant \left(\frac{L}{2\pi}\right) \frac{2}{3} \frac{1}{m} \epsilon^3 \tag{A8}$$

Proof.

$$\begin{aligned} H(\phi, \epsilon) &= \sum_{\tau} \sum_{|k|<\epsilon} \frac{(k + \tau k_F)^2}{2m} a^*(k + \tau k_F) a(k + \tau k_F) \\ &= \sum_{\tau} \sum_{|k|<\epsilon} \left(\tau \frac{k_F}{m} k + \frac{k_F^2}{2m} + \frac{k^2}{2m}\right) a_\tau^*(k) a_\tau(k) \end{aligned}$$

$$H(\phi, \epsilon) = \frac{k_F}{m} H(\psi, \epsilon) + \frac{k_F^2}{2m} Q(\epsilon) + R$$

with

$$\|R\| \leqslant \sum_{|k|<\epsilon} \frac{k^2}{2m} \leqslant \left(\frac{L}{2\pi}\right) \frac{2}{3} \frac{1}{m} \epsilon^3 \quad \blacksquare$$

Definition 9. Since one is interested in states $\Omega(c)$ that are small deviations from the Fermi sea,

$$\begin{aligned} a(k)\Omega(c) &= 0 \quad \text{for} \quad k \notin S_F \\ a^*(k)\Omega(c) &= 0 \quad \text{for} \quad k \in S_F \end{aligned} \tag{A9}$$

we define as usual new operators $c(k)$,

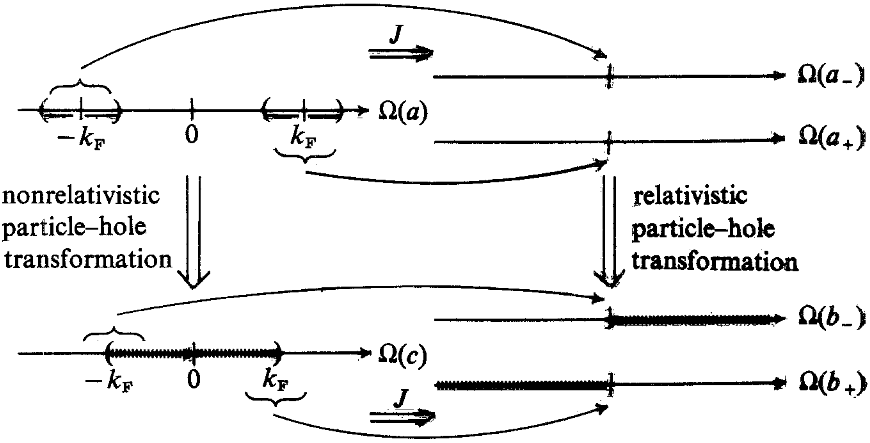
$$c(k) := \begin{cases} a(k) & \text{for} \quad k \notin S_F \\ a^*(-k) & \text{for} \quad k \in S_F \end{cases} \tag{A10}$$

and the new vacuum $\Omega(c)$, $c(k)\Omega(c) = 0$. The set S_F denotes the Fermi sphere $(-k_F, k_F)$.

Lemma 10. We define the particle [resp. hole] destruction operators $b_+(k)$ [resp. $b_-(k)$] in the Dirac formalism as

$$b_+(k) := a_{\operatorname{sign} k}(k), \qquad b_-(k) := a_{\operatorname{sign} k}(-k) \tag{A11}$$

with the vacua $\Omega(b_\tau)$; $b_\tau(k)\Omega(b_\tau) = 0$ and $\tau \in \{+, -\}$. Then the following diagram is commutative:



The simple proof will be omitted.

Corollary 11. Let $\tilde{H}_0(\phi, \epsilon)$, $\tilde{H}_0(\psi, \epsilon)$, $\tilde{Q}(\epsilon)$, $\tilde{R}(\epsilon)$, $Z(x)$, and $C(k_F, \epsilon)$ be defined as

$$\tilde{H}_0(\phi, \epsilon) = \sum_\tau \sum_{|k_F + \tau k| < \epsilon} \omega_k[1 - 2\chi_F(k)]c^*(k)c(k)$$

$$\tilde{H}_0(\psi, \epsilon) := \sum_\tau \sum_{|k| < \epsilon} |k| b_\tau^*(k) b_\tau(k)$$

$$\tilde{Q}(\epsilon) := \sum_\tau \sum_{|k| < \epsilon} \tau b_\tau^*(k) b_\tau(k)$$

$$\tilde{R}(\epsilon) := \sum_\tau \sum_{|k| < \epsilon} \tau \omega_k b_\tau^*(k) b_\tau(k) \tag{A12}$$

$$Z(x) := \frac{2\pi}{L} \frac{x}{6m} \left[2\left(\frac{L}{2\pi} x\right)^2 + 1\right]$$

$$C(k_F, \epsilon) := Z(k_F - \epsilon) - Z(k_F) + \frac{L}{2\pi} \frac{\epsilon}{3m} (\epsilon^2 - 3k_F\epsilon + 3k_F^2)$$

where the characteristic function $\chi_F(k)$ is one for $k \in S_F$ and zero otherwise. Then

$$\tilde{H}_0(\phi, \epsilon) = \frac{k_F}{m} \tilde{H}_0(\psi, \epsilon) + \frac{k_F^2}{2m} \tilde{Q}(\epsilon) + \tilde{R}(\epsilon) + C(k_F, \epsilon) \tag{A13}$$

holds.

The proof is a direct consequence of Theorem 8.

APPENDIX B. BOSE–FERMI RECIPROCITY

In order to enhance the calculational efficiency in what follows, we will use the cyclic group of order two $S := \mathbb{Z}/2\mathbb{Z}$ to label various dichotomic choices. Thus, depending on the context of its use, the identity element of S will label a particle, a creation operator, a "negative" helicity component, a field moving to the "right" (with respect to an arbitrary but fixed choice of what "negative" and "right" is). Correspondingly, the nontrivial element of S will label a hole, an annihilation operator, a "positive" helicity component, a field moving to the "left." We also adopt the convention that, when a (nonvanishing) real variable occurs in the place of an S variable, the *sign* of the real variable is to be understood, with a positive sign corresponding to the identity of S.

For the fermion creation and annihilation operators this convention means that we write

$$\alpha_+{}^+, \alpha_+{}^-, \alpha_-{}^+, \alpha_-{}^- \quad \text{for} \quad \alpha_p{}^*, \alpha_p, \alpha_h{}^*, \alpha_h$$

The irreducible Fock nature of this system is therefore expressed by

$$\{\alpha_\tau{}^\sigma(\kappa), \alpha_{\tau'}^{\sigma'}(\kappa')\} = \delta_{\sigma,-\sigma'}\, \delta_{\tau,\tau'}\, \delta_{\kappa,\kappa'} \tag{B1}$$

$$\alpha_\tau{}^-(\kappa)\Omega = 0 \tag{B2}$$

$\mathscr{D}$ is dense in $\mathscr{F}$, where $\mathscr{D}$ is the linear span of the monomials

$$\prod_j^{1,n} \alpha_{\tau_j}^+(\kappa_j)\Omega \tag{B3}$$

The definition of the time-dependent, free Fermi field corresponding to Eq. (3.2) can be rewritten as

$$\begin{pmatrix} \psi_-(v^-) \\ \psi_+(v^+) \end{pmatrix} = \frac{1}{\sqrt{L}} \sum_{\kappa \in \overline{K}_L'} \begin{pmatrix} \theta(-\kappa) & \theta(\kappa) \\ \theta(\kappa) & \theta(-\kappa) \end{pmatrix} \begin{pmatrix} e^{i(\kappa x - |\kappa| t)} & \alpha_+{}^-(\kappa) \\ e^{i(\kappa x + |\kappa| t)} & \alpha_-{}^+(-\kappa) \end{pmatrix} \tag{B4}$$

where, for $\tau \in \{\pm\} = S$, $v^\tau := x - \tau t$ and $\theta(\kappa) := \frac{1}{2}[1 + \operatorname{sign}(\kappa)]$.

In terms of $\psi_\tau{}^- \equiv \psi_\tau$ and $\psi_\tau{}^+ \equiv \psi_\tau{}^*$ this equation and its adjoint can be condensed to read

$$\psi_\tau{}^\sigma(v^\tau) = \frac{1}{\sqrt{L}} \sum_\kappa e^{-i\kappa\sigma\tau v^\tau} \alpha_\kappa^{\sigma\kappa}(\tau|\kappa|) \tag{B5}$$

The CAR relations of Eq. (B1) imply that

$$\{\psi_\tau{}^\sigma(v^\tau), \psi_{\tau'}^{\sigma'}(v'^{\tau'})\} = \delta_{\sigma,-\sigma'}\, \delta_{\tau,\tau'} \frac{1}{L} \sum_\kappa \exp[i\kappa(v^\tau - v'^{\tau'})] \tag{B6}$$

The representation of Eq. (3.8) of the charges and that of the Fourier components of the currents Eq. (3.10) take the form (on the domain $\mathscr{D}$)

$$Q_\tau = \sum_\lambda :\alpha_\lambda{}^\lambda(\tau|\lambda|)\alpha_\lambda^{-\lambda}(\tau|\lambda|): \tag{B7}$$

$$\tilde{j}_\tau(p) = \sum_{\mu-\lambda=\tau p} :\alpha_\lambda{}^\lambda(\tau|\lambda|)\alpha_\mu^{-\mu}(\tau|\mu|): ; \qquad p \neq 0 \tag{B8}$$

A careful evaluation on $\mathscr{D}$ leads to the commutation relations

$$[\tilde{j}_\tau(p), \tilde{j}_{\tau'}(p')] = (L/2\pi)\tau p\ \delta_{\tau,\tau'}\ \delta_{p,-p'} \tag{B9}$$

From this equation it follows immediately that the operators $c(p)$ and $c^*(p)$ defined (on $\mathscr{D}$) by

$$c^\sigma(p) := -i\sigma(2\pi/L|p|)^{1/2}\tilde{j}(-\sigma p); \qquad p \neq 0 \tag{B10}$$

actually satisfy the CCR relations

$$[c^\sigma(p), c^{\sigma'}(p')] = \sigma'\ \delta_{\sigma,-\sigma'}\ \delta_{p,p'}; \qquad p \neq 0 \neq p' \tag{B11}$$

For later reference we write down the Bose vacuum state for the (n_-, n_+) charge sector of $\mathscr{F}$:

$$\Omega_{n_-,n_+} := \prod_{-(2\pi/L)|n_-|<\kappa_-<0} \alpha_{n_-}^+(\kappa_-) \prod_{(2\pi/L)|n_+|>\kappa_+>0} \alpha_{n_+}^+(\kappa_+)\Omega \tag{B12}$$

where the ordering in each product is chosen such that the absolute values of κ_+ and κ_- decrease from left to right.

Turning now to the Bose field, we see that the definition of Eq. (3.14) amounts to

$$\varphi_\tau{}^\mu(v^\tau) = \sum_{p>0} (2\pi/Lp)^{1/2} e^{-i\mu\tau p v^\tau} c^\mu(\tau p) \tag{B13}$$

$$\varphi_\tau(v^\tau) = \sum_{p\neq 0} (2\pi/L|p|)^{1/2} e^{-i\tau p v^\tau} c^p(\tau|p|) \tag{B14}$$

Defining

$$\Delta_{\tau,\tau'}(\xi) := -2\tau\ \delta_{\tau,\tau'} \sum_{n=1}^{\infty} \frac{\sin[n(2\pi/L)\xi]}{n}$$

$$= \tau\ \delta_{\tau,\tau'}\left(\frac{2\pi}{L}\xi - \pi\right) \qquad \text{if} \quad 0 < \xi < L \tag{B15}$$

it follows from Eq. (B14) that

$$[\varphi_\tau(v^\tau), \varphi_{\tau'}(v'^{\tau'})] = -i\ \Delta_{\tau,\tau'}(v^\tau - v'^{\tau'}) \tag{B16}$$

If U_τ^σ denotes the unique unitary operator on $\mathscr{F}$ that maps each Ω_{n_-,n_+} to $\Omega_{n_-+\delta_{\tau,-}.\sigma,\, n_++\delta_{\tau,+}.\sigma}$ and furthermore commutes with all φ_τ^μ, then $(U_\tau^\sigma)^* = U_\tau^{-\sigma}$ and we find

$$U_\tau^\sigma Q_{\tau'} = (Q_{\tau'} - \sigma\,\delta_{\tau,\tau'})U_\tau^\sigma \tag{B17}$$

Lemma 1. The fields ψ_τ^σ and $\varphi_{\tau'}$ are related by the formula

$$[\psi_\tau^\sigma(v^\tau), \varphi_{\tau'}(v'^{\tau'})] = i\sigma\psi_\tau^\sigma(v^\tau)[\varphi_\tau(v^\tau), \varphi_{\tau'}(v'^{\tau'})] \tag{B18}$$

Proof. A straightforward but tedious calculation involving Eqs. (B8) and (B10) yields that for $p' > 0$

$$[\alpha_\lambda^{\sigma\lambda}(\tau|\lambda|), c^{\sigma'}(\tau'p')] = i(2\pi/Lp')^{1/2}\sigma\sigma'\,\delta_{\tau,\tau'}\,\alpha_{\lambda+\sigma\sigma'p'}^{\sigma\lambda+\sigma'p'}(\tau'|\sigma\lambda + \sigma'p'|) \tag{B19}$$

According to Eqs. (B5) and (B14), this implies

$$\begin{aligned}
&[\psi_\tau^\sigma(v^\tau), \varphi_{\tau'}(v'^{\tau'})]\\
&= \sum_\lambda \sum_{\substack{p'>0\\ \sigma'}} \frac{1}{\sqrt{L}}\exp(-i\lambda\sigma\tau v^\tau)\left(\frac{2\pi}{Lp'}\right)^{1/2}\\
&\quad\times [\exp(-i\sigma'\tau'p'v'^{\tau'})][\alpha_\lambda^{\sigma\lambda}(\tau|\lambda|), c^{\sigma'}(\tau'p')]\\
&= \sum_{\substack{p'>0\\ \sigma'}} \sum_\kappa i\frac{2\pi}{L^{3/2}}\sigma\sigma'\,\delta_{\tau,\tau'}\frac{1}{p'}\\
&\quad\times \{\exp[i\sigma'\tau p'(v^\tau - v'^{\tau'})]\exp(-i\sigma\tau\kappa v^\tau)\}\alpha_\kappa^{\sigma\kappa}(\tau|\kappa|)\\
&= i\sigma\psi_\tau^\sigma(v^\tau)\sum_{\substack{p'>0\\ \sigma'}} \sigma'\,\delta_{\tau,\tau'}\frac{2\pi}{Lp'}\exp[i\sigma'\tau p'(v^\tau - v'^{\tau'})]
\end{aligned} \tag{B20}$$

where in the resummation we have set $\kappa = \lambda + \sigma\sigma'p'$.

This result in turn implies Eq. (B18) in view of Eqs. (B15) and (B16).

QED

In the following we will make use of the function

$$F_\tau^\sigma(v^\tau) \coloneqq \frac{1}{\sqrt{L}}(-)^{\delta_{\tau,+}Q_-}\exp\left[-i\sigma\tau\frac{2\pi}{L}v^\tau\left(Q_\tau + \frac{\sigma}{2}\right)\right] \tag{B21}$$

which is scalar in each charge sector of $\mathscr{F}$.

The relationship between the ψ_τ^σ and φ_τ fields is made explicit by the following (free field) *bosonization formula*:

Theorem A.

$$\psi_\tau^\sigma(v^\tau) = U_\tau^\sigma F_\tau^\sigma(v^\tau):e^{i\sigma\varphi_\tau}:(v^\tau) \tag{B22}$$

Proof. For scalar $[A, B]$ the identity $[e^A, B] = e^A[A, B]$ is valid. Accordingly, we deduce that

$$\begin{aligned}[&\vdots e^{i\sigma\varphi_\tau}\vdots(v^\tau), \varphi_{\tau'}(v'^{\tau'})] \\ &= [e^{i\sigma\varphi_\tau^+}(v^\tau)e^{i\sigma\varphi_\tau^-}(v^\tau), \varphi_{\tau'}(v'^{\tau'})] \\ &= [e^{i\sigma\varphi_\tau^+}(v^\tau), \varphi_{\tau'}(v'^{\tau'})]e^{i\sigma\varphi_\tau^-}(v^\tau) + e^{i\sigma\varphi_\tau^+}(v^\tau)[e^{i\sigma\varphi_\tau^-}(v^\tau), \varphi_{\tau'}(v'^{\tau'})] \\ &= i\sigma\vdots e^{i\sigma\varphi_\tau}\vdots(v^\tau)[\varphi_\tau(v^\tau), \varphi_{\tau'}(v'^{\tau'})]\end{aligned}$$

Since

$$\vdots e^{i\sigma\varphi_\tau}\vdots^{-1} = (e^{i\sigma\varphi_\tau^+}e^{i\sigma\varphi_\tau^-})^{-1} = e^{-i\sigma\varphi_\tau^-}e^{-i\sigma\varphi_\tau^+}$$

this result implies the relation

$$[(\vdots e^{i\sigma\varphi_\tau}\vdots)^{-1}(v^\tau), \varphi_{\tau'}(v'^{\tau'})] = -i\sigma(\vdots e^{i\sigma\varphi_\tau}\vdots)^{-1}(v^\tau)[\varphi_\tau(v^\tau), \varphi_{\tau'}(v'^{\tau'})]$$

which in conjunction with Lemma 1 of this Appendix shows that the function

$$F_\tau^{\sigma}(v^\tau) \coloneqq U_\tau^{-\sigma}\psi_\tau^{\sigma}(v^\tau)(\vdots e^{i\sigma\varphi_\tau}\vdots)^{-1}(v^\tau)$$

commutes with $\varphi_{\tau'}(v'^{\tau'})$. Since, by construction, $F_\tau^{\sigma}(v^\tau)$ maps each charge sector of $\mathscr{F}$ into itself, the irreducibility of the φ fields in each charge sector[22] shows that it depends on the charge only. Its value on the (n_-, n_+) charge sector is hence given by

$$\begin{aligned}&\langle\Omega_{n_-,n_+}, F_\tau^{\sigma}(v^\tau)\Omega_{n_-,n_+}\rangle \\ &= \langle U_\tau^{\sigma}\Omega_{n_-,n_+}, \psi_\tau^{\sigma}(v^\tau)\Omega_{n_-,n_+}\rangle \\ &= \sum_\lambda (1/\sqrt{L})e^{-i\lambda\sigma\tau v^\tau}\langle\Omega_{n_-+\sigma\delta_{\tau,-},n_++\sigma\delta_{\tau,+}}, \alpha_\lambda^{\sigma\lambda}(\tau|\lambda|)\Omega_{n_+,n_-}\rangle \\ &= (1/\sqrt{L})(-)^{\delta_{\tau,+}n_-}e^{-i\sigma\tau(2\pi/L)v^\tau(n_\tau+\sigma/2)}\end{aligned}$$

where we have used the detailed form (B12) of the Bose vacua Ω_{n_-,n_+}. Since the operator form of this last expression is incorporated in Eq. (B21), we have thus deduced Eq. (B22). QED

The bosonization formula (B22) can be generalized to more than two space-time dimensions. The method used is a straightforward reduction to the two-dimensional case and will be explained for the case of four space-time dimensions.

We start from a two-component fermion field ψ that is a solution of the Weyl equation and describes a particle with positive and an antiparticle with negative helicity,

$$\psi(\mathbf{x}) = (1/L)^{3/2}\sum_{\mathbf{k}}\{[\exp(i\mathbf{kx})\}\rho(\mathbf{k})\alpha_p(\mathbf{k}) + [\exp(-i\mathbf{kx})]\rho(\mathbf{k})\alpha_h{}^*(\mathbf{k})\} \tag{B23}$$

The sum runs over **k**'s in $(K_L')^3$. The spinor ρ depends only on the direction of the momentum $\mathbf{k} = k\mathbf{e}$, $\mathbf{e} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$; a possible choice is

$$\rho(\mathbf{e}) = \begin{pmatrix} \exp(-i\varphi/2) & \cos(\vartheta/2) \\ \exp(i\varphi/2) & \sin(\vartheta/2) \end{pmatrix} \tag{B24}$$

We reorder the summation in (B23) as follows:

$$\psi(\mathbf{x}) = (1/L)^{3/2} \sum_{\mathbf{e}} \rho(\mathbf{e}) \sum_{k} (\{\exp[ik(\mathbf{ex})]\}\alpha_p(\mathbf{e}, k) + \{\exp[-ik(\mathbf{ex})]\}\alpha_h^*(\mathbf{e}, k)) \tag{B25}$$

The first sum runs over a discrete set on the 3-sphere and the second one over the positive elements in $K'_{L(\mathbf{e})}$, where $L(\mathbf{e}) = L\cos\vartheta$ is the volume cutoff of the fermion field $\psi_+(\mathbf{e}, y)$ in two space–time dimensions defined by

$$\psi_+(\mathbf{e}, y) = (1/\sqrt{L}) \sum_{k} (e^{iky}\alpha_p(\mathbf{e}, k) + e^{-iky}\alpha_h^*(\mathbf{e}, k)) \tag{B26}$$

Clearly the fermion field ψ is now a sum over the family of two-dimensional fermion fields with a spinorial weight factor,

$$\psi(\mathbf{x}) = \frac{1}{L} \sum_{\mathbf{e}} \left(\frac{L(\mathbf{e})}{L}\right)^{1/2} \rho(\mathbf{e})\psi_+(\mathbf{e}, (\mathbf{ex})) \tag{B27}$$

The $\psi_+(\mathbf{e}, y)$ anticommute for different values of **e**. They can be rewritten in terms of boson operators due to formula (B22); this leads to

$$\psi(\mathbf{x}, t) = \frac{1}{L} \sum_{\mathbf{e}} \left(\frac{L(\mathbf{e})}{L}\right)^{1/2} \rho(\mathbf{e}) U_+(\mathbf{e}) F_+(\mathbf{e}, t - (\mathbf{ex})) {:}e^{-i\varphi_+}{:}(\mathbf{e}, t - (\mathbf{ex})) \tag{B28}$$

φ_+ denotes the scalar potential of the current belonging to the corresponding two-dimensional fermion field.

For computations it might be more convenient to scale the fermion fields $\psi_+(\mathbf{e}, y)$ so as to have a cutoff parameter independent of **e**.

Finally, Kronig's identity generalizes to the higher dimensional case as

$$H_0^{\mathrm{F}}(\psi) = \sum_{\mathbf{e}} \left[H_0^{\mathrm{B}}(\varphi(\mathbf{e})) + \frac{\pi}{L(\mathbf{e})} Q_+^2(\mathbf{e})\right] \tag{B29}$$

APPENDIX C. INTERACTING FIELDS AND STATISTICAL AVERAGES

The interacting Bose and Fermi Heisenberg fields of the Luttinger model are defined as

$$\Phi_\tau^\mu(t, x) := e^{iHt} U \varphi_\tau^\mu(x) U^* e^{-iHt} \tag{C1}$$

and

$$\Psi_\tau^\sigma(t, x) := e^{iHt} U \Psi_\tau^\sigma(x) U^* e^{-iHt} \tag{C2}$$

With the aid of Eqs. (3.30) we deduce that

$$\Phi_\tau^\mu(t, x) = \sum_{q \neq 0} \left(\frac{2\pi}{L|q|}\right)^{1/2} \{\exp[i(\operatorname{sgn} q \cdot \omega^q t - \mu\tau|q|x)]\} h_q^{\mu q} c^q(\mu\tau q) \tag{C3}$$

Conjugating the bosonization formula (B22) with $e^{iHt}U$, we obtain the modified (interacting field) bosonization formula

$$\Psi_\tau^\sigma(t, x) = e^{-A} U_\tau^\sigma F_\tau^\sigma(t, x) \vdots e^{i\sigma\Phi_\tau} \vdots (t, x) \tag{C4}$$

where $\Phi_\tau = \sum_\mu \Phi_\tau^\mu$ and

$$A = \langle \Omega, \Phi_\tau^+ \Phi_\tau^- \Omega \rangle = \sum_{p>0} \frac{2\pi}{Lp} (h_p^-)^2 = \frac{\pi}{L} \sum_{p>0} \left(\frac{1}{\omega_p} - \frac{1}{p}\right) \tag{C5}$$

This last term results from the Bose normal ordering of the Φ terms and is finite by the assumption of Eq. (3.24) on the interaction.

To facilitate the writing of formulas in the computation of the thermodynamic correlation functions, we introduce the following notation. For any natural number n, if σ_j, τ_j, t_j, and x_j are given for $1 \leqslant j \leqslant n$, we define

$$J_\pm := \{j / \tau_j = \pm\} \tag{C6}$$

$$\Psi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x}) = \prod_j^{1,n} \Psi_{\tau_j}^{\sigma_j}(t_j, x_j) \tag{C7}$$

$$\Phi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x}) = \sum_j^{1,n} \sigma_j \Phi_{\tau_j}(t_j, x_j) = \sum_{\mu, q \neq 0} \epsilon_{q\boldsymbol{\tau}}^{\mu\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x}) c^\mu(q) \tag{C8}$$

$$\epsilon_{q\boldsymbol{\tau}}^{\mu\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x}) := \left(\frac{2\pi}{L|q|}\right)^{1/2} \sum_j^{1,n} \sigma_j e^{i\mu(\omega_q t_j - q x_j)} h_q^{\tau j q} \tag{C9}$$

$$Q_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x}) := \sum_j^{1,n} \frac{2\pi}{L} \sigma_j (t_j - \tau_j x_j) Q_{\tau_j} \tag{C10}$$

In order to deduce a manageable form of the bosonization formula for $\Psi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}$ we compute with the aid of Eq. (B17) that

$$\left(\prod_j^{1,n} U_{\tau_j}^{\sigma_j}\right)^{-1} \prod_j^{1,n} [U_{\tau_j}^{\sigma_j} F_{\tau_j}^{\sigma_j}(t_j, x_j)]$$

$$= \prod_j^{1,n} \frac{1}{\sqrt{L}} (-)^\Xi \exp\left[-i \frac{2\pi}{L} \sigma_j \tau_j v_j^{\tau_j} \left(Q_{\tau_j} + \sum_k^{j+1,n} \delta_{\tau_j,\tau_k} \sigma_k + \tfrac{1}{2}\sigma_j\right)\right] \tag{C11}$$

where $\Xi = \delta_{\tau_j,+}(Q_- + \sum_k^{j+1,n} \delta_{\tau_k,-} \cdot \sigma_k)$. In terms of

$$A_{\tau,\tau'}(t, x) := \sum_{q \neq 0} \frac{2\pi}{L|q|} e^{-i(\omega_q t - qx)} h_q^{\tau q} h_q^{\tau' q} \tag{C12}$$

the Bose normal ordering yields the formula

$$\prod_{j=1}^{n} :\exp(i\sigma_j \Phi_{\tau_j}):(t_j, x_j) = \exp\left[-\sum_{j<k}^{1,n} \sigma_j\sigma_k A_{\tau_j\tau_k}(t_j - t_k, x_j - x_k)\right] :\exp(i\Phi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}):(\mathbf{t}, \mathbf{x}) \quad \text{(C13)}$$

Due to the fact that H has a purely Bose expression, the vacuum expectation value and thermodynamic average of $\Psi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x})$ will be nonvanishing only if, for each τ value, the number of Ψ_τ^+ terms equals the number of Ψ_τ^- terms in this product. This happens precisely if

$$\prod_j U_{\tau j}^{\sigma j} = id_{\mathscr{F}}$$

In the following we will *assume* that the σ_j and τ_j have been chosen to satisfy these conditions. In particular, this implies that J_+ has an even number of elements and hence that

$$\prod_{j=1}^{n} (-)^{\delta\tau_j,+Q_-} = (-)^{|J_+|Q_-} = 1$$

Using Eqs. (C11) and (C13), we find that

$$\Psi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x}) = \langle \Omega, \Psi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x})\Omega\rangle \exp(iQ_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}})(\mathbf{t}, \mathbf{x}) :\exp(i\Phi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}):(\mathbf{t}, \mathbf{x}) \quad \text{(C14)}$$

and

$$\begin{aligned} \log\langle \Omega, \Psi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x})\Omega\rangle &= \sum_{j}^{1,n} \left[i\frac{\pi}{L}(t_j - \tau_j x_j) - \frac{1}{2}\log L - A \right] \\ &+ \sum_{j<k}^{1,n} \sigma_j\sigma_k \Big[i\pi\sigma_j\, \delta_{\tau_j,+}\, \delta_{\tau_k,-} + i\frac{2\pi}{L}(t_j - \tau_j x_j)\, \delta_{\tau_j,\tau_k} \\ &- A_{\tau_j,\tau_k}(t_j - t_k, x_j - x_k)\Big] \end{aligned} \quad \text{(C15)}$$

If $\beta > 0$ is the inverse temperature the thermodynamic correlation functions involve the traces (in terms of obvious notation for the $\pm$ charges)

$$\begin{aligned} \mathrm{Tr}_{\mathscr{F}}\{[\exp(-\beta H)]\Psi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x})\} &= [\exp(\beta\eta)]\langle \Omega, \Psi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}(\mathbf{t}, \mathbf{x})\Omega\rangle \\ &\times \mathrm{Tr}_+\left[\exp\left(-\beta\frac{\pi}{L}Q_+^2\right)\exp(iQ_+^{\boldsymbol{\sigma}})(\mathbf{t}, \mathbf{x})\right] \\ &\times \mathrm{Tr}_-\left[\exp\left(-\beta\frac{\pi}{L}Q_-^2\right)\exp(iQ_-^{\boldsymbol{\sigma}})(\mathbf{t}, \mathbf{x})\right] \\ &\times \mathrm{Tr}_B\left\{\exp\left[-\beta\sum_{q\neq 0}\omega_q c^+(q)c^-(q)\right] :\exp(i\Phi_{\boldsymbol{\tau}}^{\boldsymbol{\sigma}}):(\mathbf{t}, \mathbf{x})\right\} \end{aligned} \quad \text{(C16)}$$

The charge traces $\mathrm{Tr}_{\pm}$ are easily evaluated in terms of the third Jacobian theta function [see formula (16.27.3) of Ref. 35]

$$\vartheta_3(z, q) \coloneqq 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz); \qquad |q| < 1 \tag{C17}$$

to give

$$\begin{aligned}
&\mathrm{Tr}_{\pm}\left\{\exp\left(-\beta \frac{\pi}{L} Q_{\pm}{}^2\right) \exp[iQ_{\pm}{}^{\sigma}(\mathbf{t}, \mathbf{x})]\right\} \\
&= \sum_{n_{\pm} \in \mathbb{Z}} \exp\left(-\beta \frac{\pi}{L} n_{\pm}{}^2\right) \exp\left[i \sum_{j \in J_{\pm}} \sigma_j \frac{2\pi}{L}(t_j \mp x_j) n_{\pm}\right] \\
&= \vartheta_3\left(\frac{\pi}{L} \sum_{j \in J_{\pm}} \sigma_j(t_j \mp x_j), \exp\left(-\beta \frac{\pi}{L}\right)\right)
\end{aligned} \tag{C18}$$

On the other hand, due to Eq. (C8) the Bose trace is found to be an infinite product

$$\begin{aligned}
&\mathrm{Tr}_B\left\{\exp\left[-\beta \sum_{q \neq 0} \omega_q c^+(q) c^-(q)\right] \vdots \exp(i\Phi_{\tau}{}^{\sigma}) \vdots (\mathbf{t}, \mathbf{x})\right\} \\
&= \prod_{q \neq 0} \mathrm{Tr}_q\{\exp[-\beta\omega_q c^+(q)c^-(q)] \exp[i\epsilon_{q\tau}^{+\sigma}(\mathbf{t}, \mathbf{x})c^+(q)] \\
&\quad \times \exp[i\epsilon_{q\tau}^{-\sigma}(\mathbf{t}, \mathbf{x})c^-(q)]\}
\end{aligned} \tag{C19}$$

Noting that $\epsilon_{q\tau}^{-\sigma} = (\epsilon_{q\tau}^{+\sigma})^*$, a typical factor in (C19) is evaluated as follows

$$\begin{aligned}
&\mathrm{Tr}(e^{-\beta\omega c^+ c^-} e^{i\epsilon^+ c^+} e^{i\epsilon^- c^-}) \\
&= \sum_{n \geqslant 0} (n!)^{-1} e^{-\beta\omega n} \langle c^{+n}\Omega, e^{i\epsilon^+ c^+} e^{i\epsilon^- c^-} c^{+n}\Omega\rangle \\
&= \sum_{n \geqslant 0} (n!)^{-1} e^{-\beta\omega n} \langle (c^+ - i\epsilon^-)^n\Omega, (c^+ + i\epsilon^-)^n\Omega\rangle \\
&= \sum_{n \geqslant 0} \sum_{m,l}^{0,n} (n!)^{-1} e^{-\beta\omega n} \binom{n}{l}\binom{n}{m} (-i\epsilon^-)^l (i\epsilon^-)^m \langle c^{+(n-l)}\Omega, c^{+(n-m)}\Omega\rangle \\
&= \sum_{l \geqslant 0} \sum_{m \geqslant 0} (-)^m \frac{(l+m)!}{l!\, m!\, m!} e^{-\beta(l+m)\omega} |\epsilon^+|^{2m} \\
&= \sum_{l \geqslant 0} e^{-\beta\omega l}\, {}_1F_1(l+1; 1; -e^{-\beta\omega}|\epsilon^+|^2)
\end{aligned}$$

where [see formula (6.1.1) of Ref. 35]

$${}_1F_1(a; b; z) \coloneqq \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \tag{C20}$$

with $(a)_n = a\cdot(a+1)\cdots(a+n-1)$ is the confluent hypergeometric function or Kummer function.

According to formula (6.4.3) of Ref. 35,

$$ {}_1F_1(a;c;zz') = z^{-a}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\frac{(z-1)^n}{z^n}\,{}_1F_1(a+n;c;z') \tag{C21}$$

we obtain by identifying

$$a = c = 1, \qquad \frac{z-1}{z} = e^{-\beta\omega}, \qquad z' = -e^{-\beta\omega}|\epsilon^+|^2$$

that the above trace equals

$$\begin{aligned} z\,{}_1F_1(1;1;zz') &= (1-e^{-\beta\omega})^{-1}\,{}_1F_1\left(1;1;-\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}|\epsilon^+|^2\right)\\ &= (1-e^{-\beta\omega})^{-1}\exp\left(-\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}|\epsilon^+|^2\right) \end{aligned}\tag{C22}$$

since ${}_1F_1(1;1;z) = \exp(z)$.

Collecting the results in Eqs. (C18), (C19), and (C22) and inserting them into Eq. (C16), we find that

$$\begin{aligned} &\mathrm{Tr}_{\mathscr{F}}[e^{-\beta H}\Psi_\tau^\sigma(\mathbf{t},\mathbf{x})]\\ &\quad= e^{\beta\eta}\langle\Omega, \Psi_\tau^\sigma(\mathbf{t},\mathbf{x})\Omega\rangle\\ &\qquad\times \vartheta_3\left(\frac{\pi}{L}\sum_{j\in J_+}\sigma_j(t_j-x_j); e^{-\beta\pi/L}\right)\vartheta_3\left(\frac{\pi}{L}\sum_{j\in J_-}\sigma_j(t_j+x_j); e^{-\beta\pi/L}\right)\\ &\qquad\times\prod_{q\neq 0}\left[(1-e^{-\beta\omega_q})^{-1}\exp\left(-\frac{e^{-\beta\omega_q}}{1-e^{-\beta\omega_q}}|\epsilon_{q\tau}^{+\sigma}(\mathbf{t},\mathbf{x})|^2\right)\right] \end{aligned}\tag{C23}$$

As a special case of formula (C23), we find for the empty product ($n = 0$)

$$\mathrm{Tr}_{\mathscr{F}}\, e^{-\beta H} = e^{\beta\eta}[\vartheta_3(0; e^{-\beta\pi/L})]^2\prod_{q\neq 0}(1-e^{-\beta\omega_q})^{-1} \tag{C24}$$

According to formula (16.30.3) of Ref. 35, the Jacobi theta functions satisfy the identity

$$\log\frac{\vartheta_3(\alpha+\beta;q)}{\vartheta_3(\alpha-\beta;q)} = 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{(-q)^n}{1-q^{2n}}\sin(2n\alpha)\sin(2n\beta) \tag{C25}$$

Choosing $\alpha = \beta = \gamma/2$, this leads to

$$\log\frac{\vartheta_3(\gamma;q)}{\vartheta_3(0;q)} = 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{(-q)^n}{1-q^{2n}}[\sin(n\gamma)]^2 \tag{C26}$$

Using Eqs. (C23), (C24), and (C26), we finally conclude that

$$\log\langle\langle\Psi_\tau^\sigma(\mathbf{t},\mathbf{x})\rangle\rangle \coloneqq \log\{\mathrm{Tr}_{\mathscr{F}}[e^{-\beta H}\Psi_\tau^\sigma(\mathbf{t},\mathbf{x})]/\mathrm{Tr}_{\mathscr{F}}(e^{-\beta H})\}$$

$$= \log\langle\Omega, \Psi_\tau^\sigma(\mathbf{t},\mathbf{x})\Omega\rangle + 4\sum_{n=1}^{\infty}\frac{(-)^n}{n}\frac{e^{-\beta(\pi/L)n}}{1-e^{-\beta(\pi/L)n}}$$

$$\times\left[\sin^2\left(n\frac{\pi}{L}\sum_{j\in J_+}\sigma_j v_j^+\right) + \sin^2\left(n\frac{\pi}{L}\sum_{j\in J_-}\sigma_j v_j^-\right)\right]$$

$$-\sum_{q\neq 0}\frac{e^{-\beta\omega_q}}{1-e^{-\beta\omega_q}}|\epsilon_{q\tau}^{+\sigma}(\mathbf{t},\mathbf{x})|^2 \tag{C27}$$

APPENDIX D. APPLICATIONS OF THE THERMODYNAMIC *n*-POINT FUNCTION

In this appendix we apply Eq. (C27) in order to calculate the momentum distribution at finite temperature, the time-ordered, one-particle Green's function, the density correlation functions, the susceptibilities, and the pair propagator.

Convention 1. We set Boltzmann's constant k_B equal to 1. Then $\beta \coloneqq T^{-1}$ is the inverse temperature and $\langle\langle\prod_{i=1}^n \Psi_{\tau_i}^{\sigma_i}(t_i, x_i)\rangle\rangle$ ($\sigma_i, \tau_i = \pm$) denotes the thermodynamic expectation value of $\prod_{i=1}^n \Psi_{\tau_i}^{\sigma_i}(t_i, x_i)$ in the Luttinger model, i.e.,

$$\left\langle\left\langle\prod_{i=1}^{n}\Psi_{\tau_i}^{\sigma_i}(t_i, x_i)\right\rangle\right\rangle \coloneqq \mathrm{Tr}_{\mathscr{F}}\left[e^{-\beta H}\prod_{i=1}^{n}\Psi_{\tau_i}^{\sigma_i}(t_i, x_i)\right]\Big/\mathrm{Tr}_{\mathscr{F}}(e^{-\beta H}) \tag{D1}$$

Definition 2. The thermodynamic momentum distribution, i.e., the mean number of particles of momentum $q \in K_L$ at temperature β^{-1} is defined by

$$\bar{n}(q,\beta) \coloneqq \frac{1}{L}\int_0^L dx\int_0^L dx'\, e^{-iq(x-x')}\sum_\tau I_\tau(x,x',\beta) \tag{D2}$$

where

$$I_\tau(x,x',\beta) \coloneqq \langle\langle\Psi_\tau^+(0,x)\Psi_\tau^-(0,x')\rangle\rangle \tag{D3}$$

Corollary 3.

$$I_\tau(x,x',\beta) = L^{-1}\exp\left\{4\sum_{n=1}^{\infty}\frac{(-)^n}{n}\frac{e^{-\beta(\pi/L)n}}{1-e^{-\beta(2\pi/L)n}}\right.$$

$$\left.\times\left[\sin^2\left(\frac{n\pi}{L}x\right)+\sin^2\left(\frac{n\pi}{L}x'\right)\right]\right\}\frac{e^{(\pi i/L)\tau(x-x')}}{1-e^{(2\pi i/L)\tau(x-x')}}$$

$$\times\exp\left\{-\sum_{p>0}\frac{4\pi}{Lp}[1-\cos p(x-x')]\cosh 2\lambda p\,\frac{e^{-\beta\omega_p}}{1-e^{-\beta\omega_p}}\right\}$$

$$\times\exp\left\{-\sum_{p>0}\frac{4\pi}{Lp}[1-\cos p(x-x')](\sinh\lambda_p)^2\right\} \tag{D4}$$

Remark 4. (a)

$$I_\tau(x, x', \beta) = I_{-\tau}(x', x, \beta) \tag{D5}$$

(b) For $\beta \to \infty$ only the boson part of $I_\tau(x, x', \beta)$,

$$\exp\left\{-\sum_{p>0} \frac{4\pi}{Lp}[1 - \cos p(x - x')](\sinh \lambda_p)^2\right\}$$

coincides with the results obtained by Lieb and Mattis[6] and Gutfreund and Schick,[5] where the Fourier transform of $I(x, x', \beta \to \infty)$ is computed explicitly. The remaining differences are due to the charge terms.

Corollary 5. Let the time-ordered, one-particle, finite-temperature Green's function be defined as

$$G^T(t, x, \beta) \coloneqq -i \sum_\tau \langle\langle T(\Psi_\tau^-(t, x)\Psi_\tau^+(0, 0))\rangle\rangle \tag{D6}$$

where T denotes the fermion T-product; then

(a) $\langle\langle \Psi_\tau^-(t, x)\Psi_\tau^+(0, 0)\rangle\rangle$

$$\begin{aligned} &= L^{-1} \exp\left[\frac{\pi i}{L}\tau v^\tau + 4\sum_{n=1}^{\infty} \frac{(-)^n}{n} \frac{e^{-\beta(\pi/L)n}}{1 - e^{-\beta(2\pi/L)n}} \sin^2\left(\frac{n\pi}{L} v^-\right)\right] \\ &\quad \times \exp\left[-\sum_{p>0} \frac{4\pi}{Lp}\{(\cosh \lambda_p)^2[1 - \cos(\omega_p t - p\tau x)]\right. \\ &\quad \left. + (\sinh \lambda_p)^2[1 - \cos(\omega_p t + p\tau x)]\} \frac{e^{-\beta\omega_p}}{1 - e^{-\beta\omega_p}}\right] \\ &\quad \times \exp\left[\sum_{p>0} \frac{2\pi}{Lp}\{(\cosh \lambda_p)^2 e^{-i(\omega_p t - p\tau x)}\right. \\ &\quad \left. + (\sinh \lambda_p)^2 e^{-i(\omega_p t + p\tau x)} - 2(\sinh \lambda_p)^2\right]\Bigg\} \end{aligned} \tag{D7}$$

(b)

$$\begin{aligned} G^T(t, x, \beta) = -i\sum_\tau [&\theta(t)\langle\langle \Psi_\tau^-(t, x)\Psi_\tau^+(0, 0)\rangle\rangle \\ &- \theta(-t)(\langle\langle \Psi_\tau^-(t, x)\Psi_\tau^+(0, 0)\rangle\rangle)^*] \end{aligned} \tag{D8}$$

Remark 6. The free-particle, zero-temperature Green's function $G^0(t, x)$ can be derived from (D6) in the thermodynamic limit $L \to \infty$ under consideration of two boundary conditions:

(1)
$$G^0(+0, x) - G^0(-0, x) = -i\,\delta(x) \tag{D9}$$

(2) The analyticity property reads[26]: for $\operatorname{Re} t > 0$, the function $G^0(t)$ can be continued analytically from the real axis into the right lower quadrant

of the complex variable t (Re $t > 0$, Im $t < 0$), while for Re $t < 0$, $G^0(t)$ can be continued analytically from the left semiaxis into the left upper quadrant (Re $t < 0$, Im $t > 0$). From this one obtains

$$G^0(t, x) = \frac{1}{2\pi} \frac{1}{x - t + i\,\delta(t)} \tag{D10}$$

where $\delta(t) := \delta \operatorname{sign} t$, with $\delta > 0$. This is in agreement with the results obtained by other authors.[25–27,36]

Corollary 7. Let the density correlation function be defined as

$$S^{\gtrless}(x, t, \beta) := \sum_{\tau,\tau'} S^{\gtrless}_{\tau,\tau'}(x, t, \beta) \tag{D11}$$

with

$$\begin{aligned} S^{>}_{\tau\tau'}(x, t, \beta) &:= \theta(t)\langle\langle j_\tau(t, x) j_{\tau'}(0, 0)\rangle\rangle \\ S^{<}_{\tau\tau'}(x, t, \beta) &:= \theta(-t)\langle\langle j_\tau(0, 0) j_{\tau'}(t, x)\rangle\rangle \end{aligned} \tag{D12}$$

then

$$\text{(a)}\quad S^{>}_{\tau,\tau'}(x, t, \beta) = \theta(t)\Big\{\tau\tau' \frac{1}{2\pi L} \sum_{p\neq 0} |p| h_p{}^p h_p^{\tau\tau' p} \frac{1}{1 - e^{-\beta\omega_p}} \times [e^{-\beta\omega_p} e^{i\omega_p t} e^{-ip\tau x} + e^{-i\omega_p t} e^{ip\tau' x}] - \frac{1}{L^2} \frac{d}{d\beta} \vartheta_3(0, e^{-\beta(\pi/L)})\Big\} \tag{D13}$$

$$\text{(b)}\quad S^{<}_{\tau,\tau'}(x, t, \beta) = \theta(-t)\left[\frac{1}{\theta(t)} S^{>}_{\tau,\tau'}(x, t, \beta)\right]^* \tag{D14}$$

Remark 8. $S^{\gtrless}_{\tau,\tau'}(x, t, \beta)$ can be obtained either by applying the thermodynamic n-point function or by inserting $j_\tau(t, x)$ into the definition of $S^{\gtrless}(x, t, \beta)$, where $j_\tau(t, x)$ is the Heisenberg operator of the free particle current $j_\tau(v^\tau)$.

Definition 9. The Fourier transform of $S^{\gtrless}_{\tau,\tau'}(x, t, \beta)$ is defined as ($q \in K_L$)

$$S^{\gtrless}_{\tau,\tau'}(q, t, \beta) := \int_0^L dx\, e^{-iqx} S^{\gtrless}_{\tau,\tau'}(x, t, \beta) \tag{D15}$$

Corollary 10. Let $S^{\gtrless}_{\tau,\tau'}(q, t, \beta, \eta)$ for $\eta > 0$ be defined as follows:

$$\text{(a)}\quad S^{>}_{\tau,\tau'}(q, t, \beta, \eta) := \theta(t)\tau\tau' \frac{|q|}{2\pi} \frac{1}{1 - e^{-\beta\omega_q}} [h^{-\tau q}_{-\tau q} h^{-\tau' q}_{-\tau q} e^{-\beta\omega_q} e^{it(\omega_q + i\eta)} + h^{\tau' q}_{\tau' q} h^{\tau q}_{\tau' q} e^{-it(\omega_q - i\eta)}] \tag{D16}$$

(b) $$S^{>}_{\tau,\tau'}(q, t, \beta, \eta) := \theta(-t)\tau\tau' \frac{|q|}{2\pi} \frac{1}{1 - e^{-\beta\omega_q}} [h^{\tau q}_{\tau q} h^{\tau' q}_{\tau q} e^{-\beta\omega_q} e^{-it(\omega_q + i\eta)} + h^{-\tau' q}_{-\tau' q} h^{-\tau q}_{-\tau' q} e^{it(\omega_q - i\eta)}] \tag{D17}$$

Then

$$S^{\gtrless}_{\tau,\tau'}(q, t, \beta) = \lim_{\eta\to 0} S^{\gtrless}_{\tau,\tau'}(q, t, \beta, \eta)$$

Remark 11. The infinitesimal $\eta > 0$ assures the physical boundary conditions

$$\lim_{t\to+\infty} S^{>}(q, t, \beta, \eta) = 0, \qquad \lim_{t\to=-\infty} S^{<}(q, t, \beta, \eta) = 0 \tag{D18}$$

Corollary 12. Let

$$\begin{aligned} S^{>}_{\tau,\tau'}(q, \omega, \beta, \eta) &:= \int_0^{\infty} dt\, e^{i\omega t} S^{>}_{\tau,\tau'}(q, t, \beta, \eta) \\ S^{<}_{\tau,\tau'}(q, \omega, \beta, \eta) &:= \int_{-\infty}^{0} dt\, e^{i\omega t} S^{<}_{\tau,\tau'}(q, t, \beta, \eta) \end{aligned} \tag{D19}$$

be the Fourier transform of $S^{\gtrless}_{\tau,\tau'}(q, t, \beta, \eta)$; then

$$\begin{aligned} S^{>}_{\tau,\tau'}(q, \omega, \beta, \eta) &= -\tau\tau' \frac{|q|}{2\pi i} \frac{1}{1 - e^{-\beta\omega_q}} \left[\frac{h^{-\tau q}_{-\tau q} h^{-\tau' q}_{-\tau q} e^{-\beta\omega_q}}{\omega + \omega_q + i\eta} + \frac{h^{\tau' q}_{\tau' q} h^{\tau q}_{\tau' q}}{\omega - \omega_q + i\eta}\right] \\ S^{<}_{\tau,\tau'}(q, \omega, \beta, \eta) &= +\tau\tau' \frac{|q|}{2\pi i} \frac{1}{1 - e^{-\beta\omega_q}} \\ &\quad \times \left[\frac{h^{\tau q}_{\tau q} h^{\tau' q}_{\tau q} e^{-\beta\omega_q}}{\omega - \omega_q - i\eta} + \frac{h^{-\tau' q}_{-\tau' q} h^{-\tau q}_{-\tau' q}}{\omega + \omega_q - i\eta}\right] \end{aligned} \tag{D20}$$

Remark 13. For $\beta \to \infty$ our results agree with those obtained by Dover.[25] He also discussed the unphysical plasmon pole $\omega = -\omega_q + i\eta$.

Corollary 14. (a) The susceptibility $\chi(t, x)$ without scattering across the Fermi surface

$$\chi(t, x) := -i\theta(t) \sum_{\tau,\tau'} \langle\langle [j_\tau(t, x), j_{\tau'}(0, 0)] \rangle\rangle \tag{D21}$$

is given by

$$\chi(t, x) = -\theta(t) \frac{1}{\pi L} \sum_{p\neq 0} |p|(h_p{}^p - h_p^{-p})^2 \sin(\omega_p t - px) \tag{D22}$$

(b) Let $\chi(0, q)$ $(q \in K_L)$ be the static susceptibility, with $\chi(0, q) \coloneqq \lim_{\eta, \omega \to 0} \chi(\omega + i\eta, q)$ and

$$\chi(\omega + i\eta, q) \coloneqq \int_{-\infty}^{+\infty} dt \int_0^L dx\, e^{-iqx} e^{i(\omega + i\eta)t} \chi(t, x) \tag{D23}$$

Then

$$\chi(0, q) = -\frac{1}{\pi} \frac{1}{1 + (2/\pi)\tilde{v}(q)} \tag{D24}$$

holds.

Remark 15. $\chi(0, q)$ was first calculated by Lieb and Mattis.[6]

Corollary 16. The susceptibility $\chi_F(t, x)$ with scattering across the Fermi surface is defined as

$$\chi_F(t, x) \coloneqq -i\theta(t)\langle\langle[\Psi_-^{+}(t, x)\Psi_+^{-}(t, x), \Psi_+^{+}(0, 0)\Psi_-^{-}(0, 0)]\rangle\rangle \tag{D25}$$

and reads

$$\begin{aligned}\chi_F(t, x) = 2\theta(t) \operatorname{Im}\Bigg(L^{-2} \exp\left(-\frac{2\pi i}{L} t\right) \\ \times \exp\left\{4 \sum_{n=1}^{\infty} \frac{(-)^n}{n} \frac{e^{-\beta(\pi/L)n}}{1 - e^{-\beta(\pi/L)n}} \left[\sin^2\left(\frac{n\pi}{L} v^+\right) + \sin^2\left(\frac{n\pi}{L} v^-\right)\right]\right\} \\ \times \exp\left\{\sum_{p \neq 0} \frac{2\pi}{L|p|} [1 + e^{2\lambda_p}(e^{-i(\omega_p t - px)} - 1)]\right\}\Bigg)\end{aligned} \tag{D26}$$

Corollary 17. Let the pair propagator $P(t, x)$ be defined as

$$P(t, x) \coloneqq -i\theta(t)\langle\langle[\Psi_-^{-}(t, x)\Psi_+^{-}(t, x), \Psi_+^{+}(0, 0)\Psi_-^{+}(0, 0)]\rangle\rangle \tag{D27}$$

Then

$$P(t, x) = \chi_F(t, x)|_{\lambda_p \to -\lambda_p} \tag{D28}$$

Remark 18. (a) The boson parts of $\chi_F(t, x)$ and $P(t, x)$ are compatible with the results of Luther and Peschel[37] if in their formalism the parameter α is taken to be zero.

(b) The power law behavior of χ_F and P (see Ref. 37) cannot be verified until the thermodynamic limit $L \to \infty$ is performed.

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